PERGAMON

International Journal of Solids and Structures 37 (1999) 1275–1292

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# Stress–strain fields and the effectiveness shear properties for three-phase composites with imperfect interface

Yongli Wu*, Zhong Ling, Zhifa Dong

*Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, People's Republic of China*

Received 21 October 1997; received in revised form 22 September 1998

**Abstract**

Based on the 'average stress in the matrix' concept of Mori and Tanaka (Mori, T., Tanaka, K., 1973. Average stress in matrix and average elastic energy of materials with misfitting inclusion. Acta Metall. 21, 571–580) a micromechanical model is presented for the prediction of the elastic fields in coated inclusion composites with imperfect interfaces. The solutions of the effective elastic moduli for this kind of composite are also obtained. In two kinds of composites with coated particulates and fibers, respectively, the interface imperfections are taken to the assumption that the interface displacement discontinues are linearly related to interface tractions like a spring layer of vanishing thickness. The resulting effective shear modulus for each material and the stress fields in the composite are presented under a transverse shear loading situation. 

## 1. Introduction

It is well-known that the mechanical behavior of composite materials is significantly affected by the nature of the bond between the constituents. To reinforce the bond between the inclusion and matrix, a material which is a layer of coating or interphase, whose thickness depends on the chemistry and processing conditions of the inclusions, is introduced. Many investigators have studied the mechanical properties of this kind of material in the conditions of a perfect interface, such as Walpole (1978), Benveniste et al. (1989), Nemat-Nasser and Hori (1993), Nemat-Nasser et al. (1993). Due to the presence of interphase materials, the studies show that local fields in a coated inclusion are generally not uniform. Hence, for imperfect interfaces, many authors are limited to the studies of two-phase materials, e.g., Benveniste (1984, 1985), Achenbach and Zhu (1989) and Qu (1993a). On the basis of the composite sphere assemblage and the generalized self-consistent scheme models, Hashin (1991) investigated the

* Corresponding author. Fax: 0086-10-6256-1284.
*E-mail address:* wuyl@cc5.imech.ac.cn (Y. Wu).


PII: S0020-7683(98)00295-9

effect of imperfect interfaces on the mechanical properties of the composite by representing the interface imperfection as a thin compliant interphase with much lower elastic moduli.

The presented work is mainly concerned with the predictions of stress fields and the elastic properties for the three-phase materials with imperfect interface, based on the 'average stress in matrix' concept of Mori and Tanaka (1973). The results are derived from a variant of Benveniste's (1987) re-examination of Mori–Tanaka's method. This approach represents the local fields in a coated inclusion by the fields found when the coated inclusion embedded in an unbounded matrix medium subjects to the average matrix stresses (or strains) at infinity, so that the local fields in the coating and the inclusion, and in the matrix can be evaluated by using the solution of a single coated inclusion in an infinite matrix and considering the interaction between the inclusions through the average matrix stresses. The interfacial imperfections of allowing displacement discontinuity across the interface between the interphase and matrix have been introduced in the procedure. The imperfections can affect the overall elastic properties of composite material in two ways, the local fields on each inhomogeneity and the strain due to the interfacial imperfections. In Sections 2 and 3, we describe the procedure for evaluation of local fields and overall (effective) properties of composites consisting of three anisotropy phases of arbitrary geometry. In Section 4, we show the results of the composite with coated spherical and cylindrical inclusions with imperfect interface.

## 2. Stress and strain fields in three-phase composites with imperfect interface

Consider a three-phase composite material which consists of a matrix phase $m$ with modulus $L^m_{ijkl}$, inhomogeneity phase $f$ with modulus $L^f_{ijkl}$ and third phase $g$ with modulus $L^g_{ijkl}$ which represents the coating material that encapsulates each inclusion of the $f$ matrix. The elastic constitutive equations of the phases are given in forms

$$\sigma^r_{ij} = L^r_{ijkl}\varepsilon^r_{kl} \tag{1}$$

$$\varepsilon^r_{ij} = M^r_{ijkl}\sigma^r_{kl} \tag{2}$$

where $\sigma^r_{ij}$ and $\varepsilon^r_{ij}$ are stress and strain tensors, $r = f, g, m$, $L^r_{ijkl}$ and $M^r_{ijkl} = (L^r_{ijkl})^{-1}$ are the stiffness and compliance tensors.

Define the following boundary conditions applied to a large composite material body of volume $V$ and bounding surface $S$:

$$u_i(S) = \varepsilon^0_{ij}x_j, \tag{3}$$

$$\sigma_i(S) = \sigma^0_{ij}n_j, \tag{4}$$

where $u_i(S)$ and $\sigma_i(S)$ denote the displacement and traction vectors on $S$, $\varepsilon^0_{ij}$ and $\sigma^0_{ij}$ are the applied constant stress and strain fields. $n_i$ denotes the outward normal to $S$ and $x_i$ are coordinate system.

The interface between the coating and matrix is assumed to be imperfect. As Qu (1993b) assumed, a spring layer of vanishing thickness will be used to characterize the imperfect bonding. At the interface, the interfacial traction is still continuous, but the displacement discontinuity is allowed. The compliance of the interface (spring layer) is related to the traction and the jump of the displacement at the interface. The interface conditions can be expressed in the following forms:

$$\Delta\sigma_{ij}n_j \equiv \left[\sigma_{ij}(\Omega^+) - \sigma_{ij}(\Omega^-)\right]n_j = 0, \tag{5}$$

$$\Delta u_i \equiv \left[u_i(\Omega^+) - u_i(\Omega^-)\right] = \eta_{ij}\sigma_{jk}n_k, \tag{6}$$

where $u_i(\Omega^+)$ and $u_i(\Omega^-)$ are the values of $u_i(x)$ when $x$ approaches the interface from matrix and interphase, respectively, so are $\sigma_{ij}(\Omega^+)$ and $\sigma_{ij}(\Omega^-)$. The tensor $\eta_{ij} = 0$ corresponds to a perfect interface, while $\eta_{ij} \to \infty$ represents complete interface debonding. If the constitutive characterization of the interface can be described by $\alpha$ and $\beta$ which represent the compliance in normal and tangential directions of the interface, respectively. A special form of $\eta_{ij}$ may be written by

$$\eta_{ij} = \alpha\delta_{ij} + (\beta - \alpha)n_i n_j, \tag{7}$$

where $\delta_{ij}$ is the Kronecker symbol. It can be easily shown that $\alpha$ and $\beta$ represent the compliance in the tangential and normal directions of the interface, respectively, i.e.,

$$\Delta u_i(\delta_{ik} - n_i n_k) = \alpha\sigma_{ij}n_j(\delta_{ik} - n_i n_k), \tag{7a}$$

$$\Delta u_i n_j = \beta\sigma_{ij}n_i n_j. \tag{7b}$$

When $\beta = 0$, this constitutive characterization of the interface allows relative sliding between the two surface, but no separation. The free-sliding case can be achieved by setting $\alpha \to \infty$ with $\beta = 0$. Therefore, solution to the case of small $\alpha$ with $\beta = 0$ provide approximations complementary to the free-sliding interfaces. The interface between the inclusion and interphase is assumed to be perfect. The tractions and displacements at this interface are all continuous.

Consider the composite subjected to the boundary conditions (3) and present the solution for the strain fields in every phase symbolically as

$$\varepsilon^r_{ij}(x) = A^r_{ijkl}(x)\varepsilon^0_{kl}, \quad r = f, g, m \tag{8}$$

where $A^r_{ijkl}(x)$ is a fourth-order tensor whose volume averages are usually referred to as mechanical strain concentration factors. Determination of the tensor $A^r_{ijkl}(x)$ is approximately obtained in this paper by fields in a single coated inclusion which is embedded in an unbounded matrix medium and subjected to remotely applied strain $\varepsilon^m_{ij}$ which are equal to the yet unknown average strain in the matrix.

Now consider a local matrix volume $V'$ with surface $S'$ in the $V$ which surrounds a single coated inclusion. Its boundary conditions are

$$u_i(S') = \varepsilon^m_{ij}x_j \tag{9}$$

where $\varepsilon^m_{ij}$ is the unknown average matrix strain. The interface conditions are the same as (5) and (6). Thus, the solution in each part of the reinforcement phases can be written as

$$\varepsilon^r_{ij}(x) = T^r_{ijkl}(x)\varepsilon^m_{kl}, \quad r = f, g \tag{10}$$

where $T^r_{ijkl}(x)$ relates to the single inclusion in an infinite matrix and has phase volume averages $T^r_{ijkl}$.

To determine $\varepsilon^m_{ij}$ we relate the overall uniform strain $\bar{\varepsilon}_{ij}$ to the local average strain $\varepsilon^r_{ij}$ and the average imperfect interface strain $\varepsilon^t_{ij}$ and have

$$\bar{\varepsilon}_{ij} = \sum_r c_r\varepsilon^r_{ij} + \varepsilon^t_{ij} = \varepsilon^0_{ij}, \quad r = f, g, m \tag{11a}$$

$$\varepsilon_{ij}^{t}=\frac{1}{2V_{\Omega}}\int_{\Omega}\left(\Delta u_{i}n_{j}+\Delta u_{j}n_{i}\right)\,\mathrm{d}\Omega, \tag{11b}$$

where $c_r$ are the phase volume fractions, $c_f+c_g+c_m=1, \Omega$ is the outer surface of the interface. It is seen that the integral in (11) are somewhat difficult to evaluate for arbitrary interface. But they can be calculated for the spherical and cylindrical geometry in the following example. For slightly weakened interfaces, Qu (1993b) introduced a tensor $H_{ijkl}$ to evaluate the $\varepsilon_{ij}^{t}$ approximately by using the interface conditions (6) and replacing the stress distribution on $\Omega$ by its volume average $\sigma_{ij}^{*}$ within $\Omega$.

$$\varepsilon_{ij}^{t}=c_{fg}H_{ijkl}\sigma_{kl}^{*},\quad c_{fg}=c_f+c_g \tag{12}$$

where $H_{ijkl}$ is a fourth-order tensor defined by

$$H_{ijkl}=\frac{1}{4V_{\Omega}}\int_{\Omega}\left(\eta_{ik}n_{j}n_{l}+\eta_{jk}n_{i}n_{l}+\eta_{il}n_{j}n_{k}+\eta_{jl}n_{i}n_{k}\right)\,\mathrm{d}\Omega. \tag{13}$$

It is noticed that $H_{ijkl}$ depends on the interface (springer-layer) property and the geometry of the inclusion. $\sigma_{ij}^{*}$ is related to the $\varepsilon_{ij}^{m}$. We find that this approximation cannot be introduced to the case of the large compliance of the interface, and it give a reasonable estimation only for the very small compliance of the interface. However, having a close investigation to (11b), one can assume that $\varepsilon_{ij}^{t}$ have the forms

$$\varepsilon_{ij}^{t}=c_{fg}B_{ijkl}\varepsilon_{kl}^{m}, \tag{14}$$

where $B_{ijkl}$ are the strain concentration factors. Then (10) is averaged over the volume of each phase, and the results are introduced into (11). With eqn (14) we can find the unknown average matrix strain $\varepsilon_{ij}^{m}$ as

$$\varepsilon_{ij}^{m}=\left[\sum_{r}c_{r}T_{ijkl}^{r}+c_{fg}B_{ijkl}\right]^{-1}\varepsilon_{kl}^{0},\quad r=f,g,m. \tag{15}$$

Since we only consider the state of strain in the matrix in a small volume adjacent to the coated inclusion, hence it is reasonable that

$$T_{ijkl}^{m}\cong I_{ijkl}=\tfrac{1}{2}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right). \tag{16}$$

The approximate strain field for each phase $\varepsilon_{ij}^{r}$ can be yielded by substitution of (15) into (10).

A similar procedure can also be applied for the stress boundary conditions (4). In place of (8), we can write the solution for stress fields in each phase.

$$\sigma_{ij}^{r}(x)=D_{ijkl}^{r}(x)\sigma_{kl}^{0},\quad r=f,g,m \tag{17}$$

where $D_{ijkl}^{r}(x)$ have volume phase average $D_{ijkl}$ which are referred to as the mechanical stress concentration factors. In the volume $V'$ with surface $S'$ which embedded a single coated inclusion, the solution can be assumed to be the form

$$\sigma_{ij}^{r}(x)=W_{ijkl}^{r}(x)\sigma_{kl}^{m},\ r=f,g. \tag{18}$$

The tensor $W_{ijkl}^{r}(x)$ is related to the counterpart in (10) by

$$W_{ijkl}^{r}(x)=L_{ijpq}^{r}T_{pqst}^{r}(x)M_{stkl}^{m},\ r=f,g,m \tag{19}$$

and according to the same reason in (16) we have

$$W^r_{ijkl} \cong I_{ijkl} = \tfrac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{20}$$

To determine $\sigma^m_{ij}$ we write the overall uniform stress $\bar{\sigma}_{ij}$ in term of the local average stress $\sigma^r_{ij}$

$$\bar{\sigma}_{ij} = \sum_r c_r \sigma^r_{ij} \sigma_{r_{ij}} = \sigma^0_{ij},\ r = f, g, m \tag{21}$$

and use (19) to get the result

$$\sigma^m_{ij} = \left[\sum_r c_r W^r_{ijkl}\right]^{-1} \sigma^0_{kl}. \tag{22}$$

Introducing (22) into (18), one obtain the approximate stress fields in each phase $\sigma^r_{ij}(x)$.

## 3. Effective mechanical properties

Define the elastic constitutive relations of the composite as

$$\bar{\sigma}_{ij} = L_{ijkl}\bar{\varepsilon}_{kl}, \tag{23a}$$

$$\bar{\varepsilon}_{ij} = M_{ijkl}\bar{\sigma}_{kl}, \tag{24}$$

where $\bar{\sigma}_{ij}$ and $\bar{\varepsilon}_{ij}$ denote representative volume averages stress and strain tensors, $L_{ijkl}$ and $M_{ijkl}$ are the stiffness and compliance tensors of the composite.

To determine $L_{ijkl}$, note the solution of strain fields in each phase and (1), (11), (15) under the boundary conditions (3), we have

$$\bar{\sigma}_{ij} = \sum_r c_r L^r_{ijpq} T^r_{pqkl} \varepsilon^m_{kl} = \left(\sum_r c_r L^r_{ijpq} T^r_{pqkl}\right)\left(\sum_r c_r T^r_{klsn} + c_{fg} B_{klsn}\right)^{-1} \varepsilon^0_{sn}. \tag{23b}$$

From (23a) and (23b) one gets

$$L_{ijkl} = \left(\sum_r c_r L^r_{ijpq} T^r_{pqsn}\right)\left(\sum_r c_r T^r_{snkl} + c_{fg} B_{snkl}\right)^{-1}. \tag{25}$$

Following the similar procedures under boundary conditions (4) and using (2), (11), (19), (21) and (22) one can find

$$\bar{\sigma}_{ij} = \sum_r c_r \sigma^r_{ij} = \sum_r c_r W^r_{ijkl} \sigma^m_{kl} = \sigma^0_{ij}$$

$$\bar{\varepsilon}_{ij} = \sum_r c_r \varepsilon^r_{ij} + \varepsilon^t_{ij}.$$

The average imperfect interface strains $\varepsilon^t_{ij}$ can be assumed to be

$$\varepsilon_{ij}^{t} = c_{fg} C_{ijkl} \sigma_{kl}^{m}.$$

The tensor $C_{ijkl}$ is related to the counterparts in (14) by

$$B_{ijpq} M_{pqkl}^{m} = C_{ijkl}. \tag{26}$$

Then we have

$$\bar{\varepsilon}_{ij} = \sum_{r} c_r M_{ijpq}^{r} W_{pqkl}^{r} \sigma_{kl}^{m} + c_{fg} C_{ijkl} \sigma_{kl}^{m}, \; r = f, g, m.$$

From (24) we get

$$M_{ijkl} = \left( \sum_{r} c_r M_{ijpq}^{r} W_{pqkl}^{r} + c_{fg} C_{ijkl} \right) \left( \sum_{r} c_r W_{ijkl}^{r} \right)^{-1}, \; r = f, g, m. \tag{27}$$

Recall that $M_{ijkl}$ in (27) in $L_{ijkl}$ in (25) are derived under different boundary conditions. The consistency of the methods which require the relations

$$M_{ijkl} = L_{ijkl}^{-1}, \tag{28}$$

should be satisfied

Rearrange (27) to get

$$M_{ijpq} \left( \sum_{r} c_r W_{pqkl}^{r} \right) = \left( \sum_{r} c_r M_{ijst}^{r} W_{stkl}^{r} + c_{fg} C_{ijkl} \right), \; r = f, g, m. \tag{29}$$

Using (19), (26) and (29) leads to

$$M_{ijpq} \left( \sum_{r} c_r L_{pqst}^{r} T_{stkl}^{r} \right) = \left( \sum_{r} c_r T_{ijkl}^{r} + c_{fg} B_{ijkl} \right), \; r = f, g, m. \tag{30}$$

From (25) and (30) it can see that (28) is satisfied.

It is noted that the above determination of the effective mechanical properties is based on the assumption that the average imperfect interface strain $\varepsilon_{ij}^{t}$ can be explicitly expressed by $\varepsilon_{ij}^{m}$ or $\sigma_{ij}^{m}$. However, due to the fact that the displacement discontinuities at the interfaces are allowed, the strain $\varepsilon_{ij}^{t}$ is generally implicitly involved by $\varepsilon_{ij}^{m}$ or $\sigma_{ij}^{m}$ for the arbitrary inclusion geometry, so that dual procedures described above may not always yield the physically expected result that $M_{ijkl}$ is the inverse of $L_{ijkl}$. A detail argument refers to the case of a laminated composite with debonding (Benveniste, 1984).

## 4. Examples

### *4.1. Stress fields and effective transverse modulus for the composite with coated spherical inclusions*

We now consider the problem that the composite reinforced by coated spherical particles is subjected to traction boundary conditions. The three distinct phases are assumed to be isotropic. First, consider the auxiliary problem of a single coated sphere embedded in the infinite matrix subjected to the remote pure shear tractions. The state of the remote tractions is

$$\sigma_{xx}|_{r\to\infty} = \sigma_0, \quad \sigma_{yy}|_{r\to\infty} = -\sigma_0, \quad \sigma_{zz}|_{r\to\infty} = 0. \tag{31}$$

Following Christensen and Lo (1979), the displacement fields have the form

$$u_r = U_r \sin\theta \cos 2\phi$$

$$u_\theta = U_\theta \sin\theta \cos\theta \cos 2\phi$$

$$u_\phi = U_\phi \sin\theta \sin 2\phi \tag{32}$$

where $(r, \theta, \phi)$ are the usual spherical polar coordinates and $U_r$,$U_\theta$ and $U_\phi$ are functions of $r$ only. For each of the three phases, $U_r$,$U_\theta$ and $U_\phi$ have the following form, in the inclusion phase,

$$\left.\begin{aligned} U_r^f &= A_1 r - \frac{6v_f}{1-2v_f} A_2 r^3 \\ U_\phi^f &= A_1 r - \frac{7-4v_f}{1-2v_f} A_2 r^3 \\ U_\phi^f &= -U_\phi^f \end{aligned}\right\}, \quad 0 \leqslant r \leqslant a \tag{33}$$

in the interphase,

$$\left.\begin{aligned} U_r^g &= B_1 r - \frac{6v_g}{1-2v_g} B_2 r^3 + \frac{3}{r^4} B_3 + \frac{5-4v_g}{1-2v_g}\frac{1}{r^2} B_4 \\ U_\theta^g &= B_1 r - \frac{7-4v_g}{1-2v_g} B_2 r^3 - \frac{2}{r^4} B_3 + \frac{2}{r^2} B_4 \\ U_\phi^g &= -U_\theta^g \end{aligned}\right\} \quad a \leqslant r \leqslant b \tag{34}$$

in the matrix phase,

$$\left.\begin{aligned} U_r^m &= D_1 r + \frac{3}{r^4} D_3 + \frac{5-4v_m}{1-2v_m}\frac{1}{r^2} D_4 \\ U_\theta^m &= D_1 r - \frac{2}{r^4} D_3 + \frac{2}{r^2} D_4 \\ U_\phi^m &= -U_\theta^m \end{aligned}\right\} \quad r \geq b \tag{35}$$

where $v_f$,$v_g$,$v_m$ are the Poisson's ratio of each phase, respectively, and $a$ and $b$ are outer radii of the inclusion and interphase. The $A$'s, $B$'s and $D$'s are the usual constants to be determined from the stress and displacement conditions at the two interface. The interface conditions are

$$\left.\begin{aligned} &U_r^f(a) = U_r^g(a) \\ &U_\theta^f(a) = U_\theta^g(a) \\ &\sigma_{rr}^f(a) = \sigma_{rr}^g(a) \\ &\sigma_{r\theta}^f(a) = \sigma_{r\theta}^g(a) \\ &U_r^m(b) - U_r^g(b) = \alpha\sigma_{rr}^m(b) \\ &U_\theta^m(b) - U_\theta^g(b) = \beta\sigma_{r\theta}^m(b) \\ &\sigma_{rr}^m(b) = \sigma_{rr}^g(b) \\ &\sigma_{r\theta}^m(b) = \sigma_{r\theta}^g(b) \end{aligned}\right\} \tag{36}$$

where $\alpha$ and $\beta$ are defined in (7). Introducing $\bar{A}_1 = A_1$, $\bar{A}_2 = a^2A_2$, $\bar{B}_1 = B_1$, $\bar{B}_2 = b^2B_2$, $\bar{B}_3 = B_3/b^5$, $\bar{B}_4 = B_4/b^3$, $\bar{D}_3 = D_3/b^5$, $\bar{D}_4 = D_4/b^3$, $c = a/b$ and the interface imperfect parameters $\alpha_1 = 2\alpha\mu_m/b$, $\beta_1 = 2\beta\mu_m/b$, and substituting (33)–(35) into (36), we can write the interface conditions in the matrix form as follows

$$E_{4\times4}\begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \\ \bar{B}_4 \end{bmatrix} = F_{4\times2}\begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \end{bmatrix} \tag{37}$$

where

$$E_{4\times4} = \begin{bmatrix} 1 & -6\nu_g c^2/(1-2\nu_g) & 3/c^5 & (5-4\nu_g)/\left[(1-2\nu_g)c^3\right] \\ 1 & -(7-4\nu_g)c^2/(1-2\nu_g) & -2/c^5 & 2/c^3 \\ 1 & 3\nu_g c^2/(1-2\nu_g) & -12/c^5 & -2(5-\nu)/\left[(1-2\nu_g)c^3\right] \\ 1 & -(7+2\nu_g)c^2/(1-2\nu_g) & 8/c^5 & 2(1+\nu_g)/\left[(1-2\nu_g)c^3\right] \end{bmatrix}$$

and

$$F_{4\times2} = \begin{bmatrix} 1 & -6\nu_f/(1-2\nu_f) \\ 1 & -(7-4\nu_f)/(1-2\nu_f) \\ \mu_f/\mu_g & 3\nu_f\mu_f/\left[(1-2\nu_f)\mu_g\right] \\ \mu_f/\mu_g & -(7+2\nu_f)\mu_f/\left[(1-2\nu_f)\mu_g\right] \end{bmatrix}$$

$$G_{2\times4} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \\ \bar{B}_4 \end{bmatrix} = P_{2\times2}\begin{bmatrix} \bar{D}_3 \\ \bar{D}_4 \end{bmatrix} + N_{2\times1}D_1 \tag{38}$$

where

$$G_{2\times4}=\begin{bmatrix}1 & -(7+2v_g)/(1-2v_g) & 8 & 2(1+v_g)/(1-2v_g)\\ -1 & (7-4v_g)/(1-2v_g) & 2 & -2\end{bmatrix},$$

$$P_{2\times2}=\begin{bmatrix}8\mu_m/\mu_g & 2(1+v_m)\mu_m/[(1-2v_m)\mu_g]\\ 8\beta_1+2 & -2(1+v_m)\beta_1/(1-2v_m)-2\end{bmatrix},\quad N=\begin{bmatrix}\mu_m/\mu_g\\ \beta_1-1\end{bmatrix}.$$

$$K_{2\times4}\begin{bmatrix}\bar{B}_1\\ \bar{B}_2\\ \bar{B}_3\\ \bar{B}_4\end{bmatrix}=L_{2\times2}\begin{bmatrix}\bar{D}_3\\ \bar{D}_4\end{bmatrix} \tag{39}$$

where

$$K_{2\times4}=\begin{bmatrix}0 & 2(7+5v_g)/(1-2v_g) & -40 & -24/(1-2v_g)\\ \alpha_1-\beta_1 & 6v_g(\beta_1-1)-(7-4v_g)(\alpha_1-1)/(1-2v_g) & 5-3\beta_1-2\alpha_1 & (5-4v_g)(1-\beta_1)/(1-2v_g)+2(\alpha_1-1)\end{bmatrix}$$

and

$$L_{2\times2}=\begin{bmatrix}-40\mu_m/\mu_g & -24\mu_m/[(1-2v_m)\mu_g]\\ 3(4\alpha_1+1)(1-\beta_1)+2(4\beta_1+1)(1-\alpha_1) & [2\alpha_1(5-v_m)+5-4v_m](1-\beta_1)/(1-2v_m)+2[(1+v_m)\beta_1/(1-2v_m)-1](1-\alpha_1)\end{bmatrix}.$$

Using the condition (31), when $r\to\infty$,

$$2\mu_m D_1=\sigma_0,\quad D_1=\sigma_0/2\mu_m. \tag{40}$$

Combined with (37)–(40), the constants can be determined as follows,

$$\begin{bmatrix}\bar{A}_1\\ \bar{A}_2\end{bmatrix}=R_{2\times2}^{-1}N_{2\times1}D_1=\begin{bmatrix}a_1\\ a_2\end{bmatrix}\sigma_0/2\mu_m \tag{41}$$

$$\begin{bmatrix}\bar{B}_1\\ \bar{B}_2\\ \bar{B}_3\\ \bar{B}_4\end{bmatrix}=E_{4\times4}F_{4\times2}\begin{bmatrix}\bar{A}_1\\ \bar{A}_2\end{bmatrix}=\begin{bmatrix}b_1\\ b_2\\ b_3\\ b_4\end{bmatrix}\sigma_0/2\mu_m \tag{42}$$

$$\begin{bmatrix} \bar{D}_3 \\ \bar{D}_4 \end{bmatrix} = L_{2\times 2}^{-1} K_{2\times 4} E_{4\times 4}^{-1} F_{4\times 2} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \sigma_0 / 2\mu_m \tag{43}$$

where

$$R_{2\times 2} = \left(G_{2\times 4} - P_{2\times 2} L_{2\times 2}^{-1} K_{2\times 4}\right) E_{4\times 4}^{-1} F_{4\times 2}. \tag{44}$$

All the constants are dependent upon the remote stress $\sigma_0$. We now estimate the average stresses in each phase. This can be implemented from the integral as follows,

$$\bar{\sigma}_{ij}^{l} = \frac{3}{4\pi\left(R_2^3 - R_1^3\right)} \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \int_{R_1}^{R_2} \sigma_{ij}^{l} r^2 \, \mathrm{d}r, \quad l = f, g, m \tag{45}$$

where

$$R_1 = 0, \quad R_2 = a, \quad \text{for } l = f$$

$$R_1 = a, \quad R_2 = b, \quad \text{for } l = g$$

$$R_1 = b, \quad R_2 = \infty, \quad \text{for } l = m.$$

After some length mathematics, we found the stress components in Cartesian coordinates for each phase

$$\bar{\sigma}_{xx}^{f} = \frac{\mu_f}{\mu_m} \left[ a_1 - \frac{21}{5\left(1 - 2v_f\right)} a_2 \right] \sigma_0 = W_f \sigma_0$$

$$\bar{\sigma}_{xx}^{g} = \frac{\mu_g}{\mu_m} \left[ b_1 - \frac{21}{5\left(1 - 2v_g\right)} b_2 \frac{1 - c^5}{1 - c^3} \right] = W_g \sigma_0$$

$$\bar{\sigma}_{xx}^{m} = \sigma_0$$

$$\bar{\sigma}_{yy}^{l} = -\bar{\sigma}_{xx}^{l}, \; l = f, g, m. \tag{46}$$

Other stress components are equal to zero.

Note that

$$\sigma_{xx} = \sigma_0. \tag{47}$$

From (22) we obtain

$$\sigma_{xx}^{m} = \left(c_m + c_f W_f + c_g W_g\right)^{-1} \sigma_0. \tag{48}$$

Let $\sigma_{xx}^{m}$ take the place of the traction $\sigma_0$ in the auxiliary problem, then the approximate stress fields in the coated spherical composite can be given by the same procedure as above.

Consider the average imperfect strain $\varepsilon_{ij}^{t}$ and use the integral (11) we find that

$$\varepsilon_{xx}^{t} = c_{fg}(2\Delta U_r + 3\Delta U_\theta)/5b = c_{fg} W_t \sigma_0, \; c_{fg} = c_f + c_g$$

$$\varepsilon^{t}_{yy} = -\varepsilon^{rt}_{xx}, \text{ others} = 0. \tag{49}$$

where $\Delta U_i = U^m_i - U^g_i$ $(i = r, \theta)$ are the displacement jumps at $r = b$. The effective transverse shear modulus $\mu_T$ then follows from

$$1/\mu_r = \left(c_m/\mu_m + c_f W_f/\mu_f + c_g W_g/\mu_g + 2c_{fg} W_t\right)\left(c_m + c_f W_f + c_g W_g\right)^{-1}. \tag{50}$$

### 4.2. Stress field and effective transverse shear modulus for a coated cylinder model

Let's consider the composite reinforced by coated cylindrical fibers of a circular cross-section. We assume that fiber are aligned and distributed in the matrix in a statistically homogeneous manner and each of its three distinct phases is isotropic.

Consider first the auxiliary problem of a single coated fiber embedded in the infinite matrix. The composite is subjected to traction boundary conditions. The state of the remote traction is

$$\sigma_{xx}|_{r\to\infty} = \sigma_0, \quad \sigma_{yy}|_{r\to\infty} = -\sigma_0, \quad \sigma_{zz}|_{r\to\infty} = 0. \tag{51}$$

We can use the existing solution found by Christensen and Lo (1979). The displacement fields in each of three phases are

$$\left.\begin{aligned} u^l_r &= U^l_r \cos 2\theta \\ u^l_\theta &= U^l_\theta \sin 2\theta \end{aligned}\right\} l = f, g, m. \tag{52}$$

in the inclusion phase,

$$\left.\begin{aligned} u^f_r &= \frac{b\sigma_0}{4\mu_f}\left[a_1(\eta_f - 3)\left(\frac{r}{b}\right)^3 + d_1\frac{r}{b}\right]\cos 2\theta \\ u^f_\theta &= \frac{b\sigma_0}{4\mu_f}\left[a_1(\eta_f + 3)\left(\frac{r}{b}\right)^3 + d_1\frac{r}{b}\right]\sin 2\theta \end{aligned}\right\} \quad 0 \leqslant r \leqslant a \tag{53}$$

in the coating phase,

$$\left.\begin{aligned} u^g_r &= \frac{b\sigma_0}{4\mu_g}\left[a_2(\eta_g - 3)\left(\frac{r}{b}\right)^3 + d_2\frac{r}{b} + c_2(\eta_g + 1)\frac{b}{r} + b_2\left(\frac{b}{r}\right)^3\right]\cos 2\theta \\ u^g_\theta &= \frac{b\sigma_0}{4\mu_g}\left[a_2(\eta_g + 3)\left(\frac{r}{b}\right)^3 - d_2\frac{r}{b} - c_2(\eta_g - 1)\frac{b}{r} + b_2\left(\frac{b}{r}\right)^3\right]\sin 2\theta \end{aligned}\right\} \quad a \leqslant r \leqslant b \tag{54}$$

in the matrix phase,

$$\left.\begin{aligned} u^m_r &= \frac{b\sigma_0}{4\mu_m}\left[2\frac{r}{b} + a_3(\eta_m + 1)\frac{b}{r} + c_3\left(\frac{b}{r}\right)^3\right]\cos 2\theta \\ u^m_\theta &= \frac{b\sigma_0}{4\mu_m}\left[-2\frac{r}{b} - a_3(\eta_m - 1)\frac{b}{r} + c_3\left(\frac{b}{r}\right)^3\right] sin\ 2\theta \end{aligned}\right\} r \geqslant b \tag{55}$$

and all

$$u_z^l = 0,\ l = f, g, m$$

where $a$ and $b$ are the inner and outer radii of the coating, respectively, $a_1, d_1, a_2, d_2, c_2, b_2, a_3$ and $c_3$ are unknown constants to be determined, $\mu_l$ are phase shear moduli, $r, \theta, z$ are the usual cylindrical coordinates and

$$\eta_l = 1 + 2\mu_l^{23}/K_l^{23},\ l = f, g, m$$

where $\mu_l^{23}$ is the transverse shear modulus and $K_l^{23}$ is the plane strain bulk modulus (see Christensen and Lo, 1986).

The interface conditions have the same forms as (36) and can also be represented by the matrix forms as follows. From the first four equations in (36), one can give

$$E_{4\times 4}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} + F_{4\times 2}\begin{bmatrix} a_1 \\ d_1 \end{bmatrix} = 0 \tag{56}$$

where

$$E_{4\times 4} = \begin{bmatrix} -1 & 1/c^6 & 1/c^4 & 0 \\ 0 & 3/c^4 & 4/c^2 & -1 \\ \mu_f(\eta_g - 3) & (\mu_f + \mu_g\eta_f)/c^6 & [\mu_f(\eta_g + 1) & (\mu_f - \mu_g)/c^2 \\ -\mu_g(\eta_f - 3) & & +\mu_g(\eta_f + 1)]/c^4 & \\ \mu_f(\eta_g + 3) & (\mu_f + \mu_g\eta_f)/c^6 & -[\mu_f(\eta_g - 1) & -(\mu_f - \mu_g)/c^2 \\ -\mu_g(\eta_f + 3) & & -\mu_g(\eta_f - 1)]/c^4 & \end{bmatrix}$$

$$F_{4\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } c = a/b.$$

From the last four equations in (36), one can get

$$K_{2\times 2}\begin{bmatrix} a_3 \\ c_3 \end{bmatrix} = Q_{2\times 4}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}, \tag{57}$$

$$2N_{2\times 2} + U_{2\times 2}\begin{bmatrix} a_3 \\ c_3 \end{bmatrix} + V_{2\times 4}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = 0, \tag{58}$$

where

$$K_{2\times 2} = \begin{bmatrix} -1 & -1 \\ 1+\eta_m(\alpha_1-\beta_1)+3\alpha_1+\beta_1-12\alpha_1\beta_1 & 1+2(\alpha_1+\beta_1)-12\alpha_1\beta_1 \end{bmatrix},$$

$$Q_{2\times 4} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ \mu_m[-3(\alpha_1-\beta_1) & \frac{v_m}{\mu_g}(1-\alpha_1-\beta_1) & \mu_m[\eta_g(\alpha_1-\beta_1) & \frac{\mu_m}{\mu_g}(\alpha_1-\beta_1) \\ +\eta_g(1-\alpha_1-\beta_1)]/\mu_g & & +1-\alpha_1-\beta_1)]/\mu_g & \end{bmatrix},$$

$$N_{2\times 1} = \begin{bmatrix} 1 \\ 1-2\beta_1 \end{bmatrix}, U_{2\times 2} = \begin{bmatrix} 2 & 3 \\ \eta_m-1-4\beta_1 & -(1+6\beta_1) \end{bmatrix},$$

$$V_{2\times 4} = \begin{bmatrix} 6 & -3 & -2 & -1 \\ \mu_m(\eta_g+3)/\mu_g & \mu_m/\mu_g & \mu_m(1-\eta_g)/\mu_g & -\mu_m/\mu_g \end{bmatrix}$$

where $\alpha_1 = \mu_m\alpha/b$, $\beta_1 = \mu_m\beta/b$ are the interface imperfect parameters. The constants can be determined

$$\begin{bmatrix} a_1 \\ d_1 \end{bmatrix} = 2R_{2\times 2}^{-1}N_{2\times 1}, \tag{59a}$$

$$\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = -E_{4\times 4}^{-1}F_{4\times 2}\begin{bmatrix} a_1 \\ d_1 \end{bmatrix}, \tag{59b}$$

$$\begin{bmatrix} a_3 \\ d_3 \end{bmatrix} = K_{2\times 2}^{-1}Q_{2\times 4}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \tag{59c}$$

where

$$R_{2\times 2} = 2\left(V_{2\times 4} + U_{2\times 2}K_{2\times 2}^{-1}Q_{2\times 4}\right)E_{4\times 4}^{-1}F_{4\times 2}.$$

We now estimate the average stresses in each phase as previously done. After lengthy mathematics, we found, in Cartesian coordinates, in each phase,

$$\bar{\sigma}^{f}_{xx} = \tfrac{1}{2}\left(d_1 - 3a_1c^2\right)\sigma_0 = W_f\sigma_0, \tag{60a}$$

$$\bar{\sigma}^{g}_{xx} = \frac{1}{2}\left(d_2 - 3a_2\frac{1-c^4}{1-c^2}\right)\sigma_0 = W_g\sigma_0, \tag{60b}$$

$$\bar{\sigma}^{m}_{xx} = \sigma_0, \tag{60c}$$

$$\bar{\sigma}^{l}_{yy} = -\bar{\sigma}^{l}_{xx}, \quad l = f, g, m, \quad \text{others} = 0. \tag{60d}$$

Using (22), note that

$$\sigma_{xx} = \sigma_0 \tag{61}$$

to obtain

$$\sigma^{m}_{xx} = \left(c_m + c_fW_f + c_gW_g\right)^{-1}\sigma_0. \tag{62}$$

Substitute $\pm\sigma_0 = \pm\sigma^{m}_{xx}$ into the auxiliary problem, then the approximate stress field in coated fiber composition can be given by the same procedure as above.

Consider the average imperfect interface strain $\bar{\varepsilon}^{t}_{ij}$ and use the integral (11b), we find that

$$\bar{\varepsilon}^{t}_{xx} = c_{fg}(\Delta U_r - \Delta U_0)/2b = c_{fg}W_t\sigma_0, \quad c_{fg} = c_f + c_g$$

$$\bar{\varepsilon}^{t}_{yy} = -\bar{\varepsilon}^{t}_{xx}, \quad \text{others} = 0 \tag{63}$$

where $\Delta U_i = U^{m}_{i} - U^{g}_{i} (i = r, \theta)$ are the displacement jumps at $r = b$. The effective transverse shear modulus $\mu_T$ then follows form

$$\frac{1}{2\mu_T} = \left(\frac{c_m}{2\mu_m} + \frac{c_f}{2\mu_f}W_f + \frac{c_g}{2\mu_g}W_g + c_{fg}W_t\right)\left(c_m + c_fW_f + c_gW_g\right)^{-1}. \tag{64}$$

### 4.3. Numerical results and discussion

In order to illustrate the effect of imperfect interface on the coated inclusion composite, we consider the composite consisting of the SiC inclusions, phase $f$, a carbon coating, phase $g$, embedded in Titanium aluminate matrix, phase $m$ for the spherical and cylindrical models, respectively. The phase properties are (from Benveniste et al. 1989),

$$\mu_f = 172.0\ \text{Gpa}, \quad v_f = 0.253, \quad c_f = 0.4,$$

$$\mu_g = 14.34\ \text{Gpa}, \quad v_g = 0.2, \quad c_g = 0.01,$$

$$\mu_m = 37.10\ \text{Gpa}, \quad v_m = 0.3, \quad c_m = 0.59.$$

$a/b$ =0.9918 for the sphere particle, $a/b$ =0.9877 for the fiber.

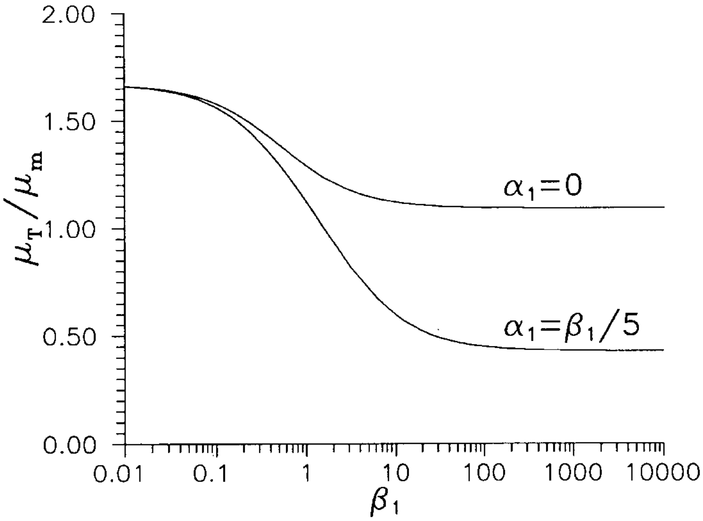


Fig. 1. Variation of effective shear modulus with shear bond parameter for spherical model.

Figures 1 and 2 show the effect of bonding on effective shear modulus for the spherical particle and cylindrical fiber cases, respectively. In one set of plots $a_1 = 0$ which denotes the perfect normal bond and imperfect shear bond. In other set of plots $\alpha_1 = \beta_1/5$ which presents the case of imperfect normal and shear bond. Because the curves for other ratio $\alpha_1/\beta_1$ have a similar trend. Therefore, we give only the typical one. It is clearly seen that the feature of normal bond has a strong and similar effect on degradation of shear modulus for the two kinds of composites. It can be seen that the $\mu_T/\mu_m$ for $\alpha_1 = 0$ and $\beta_1 = 0.01$ are 1.66 and 1.55. The numerical results obtained on the approach presented by Christensen and Lo (1979) for perfect bonding cases are 1.67 and 1.56. Our results agree with those.

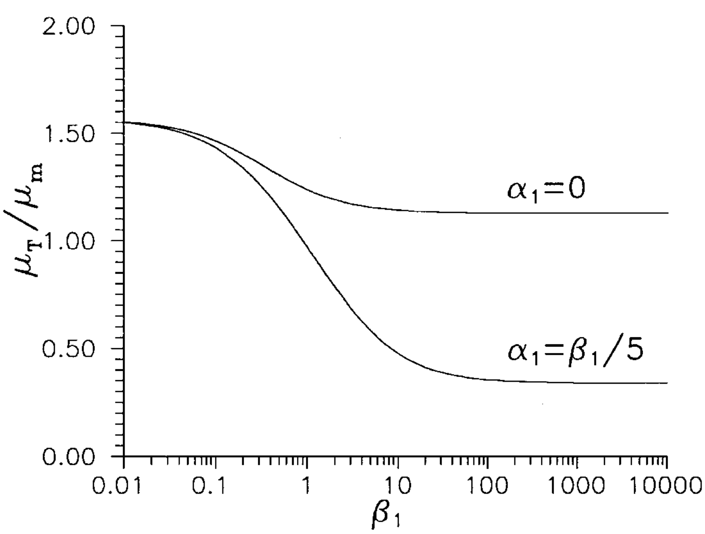


Fig. 2. Variation of effective shear modulus with shear bond parameter for cylindrical model.

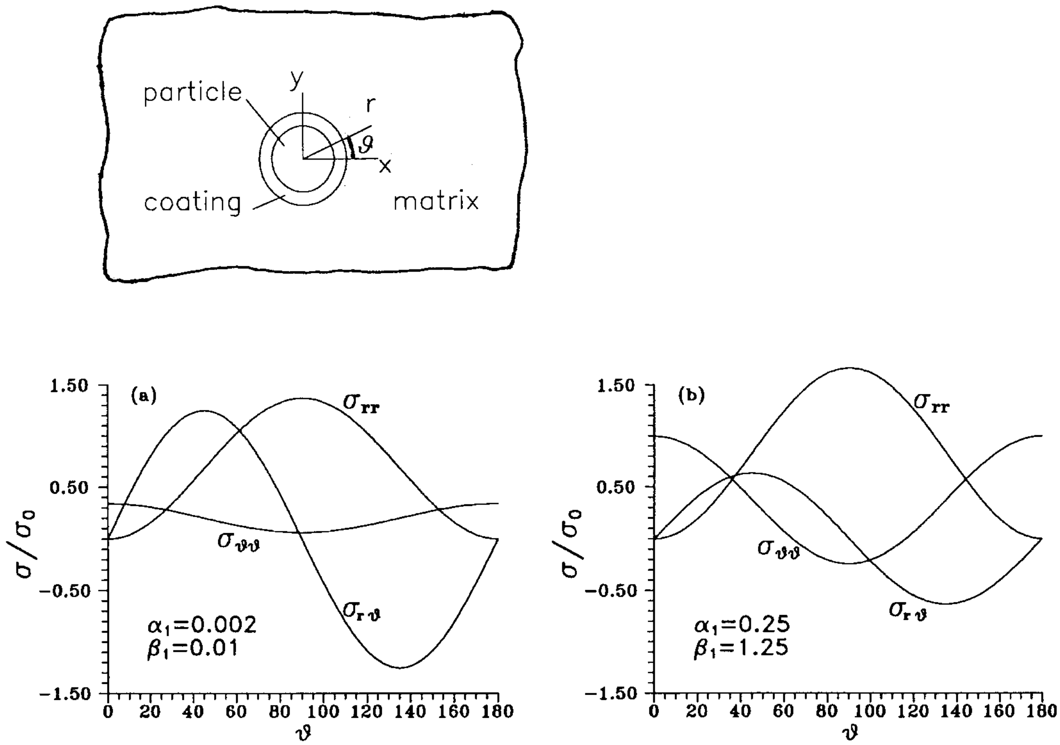


Fig. 3. Stress distributions along the interface between coating and matrix for the shear loading in XY-plane.

Figures 3 and 4 show the stress distributions along the interface between the interphase and the matrix for the coated particulate and fiber reinforced composites with different imperfect parameters, respectively. In the coated particulate composite, as the imperfect parameters $\alpha_1$ and $\beta_1$ increase, the amplitude of the stresses $\sigma_{rr}$ and $\sigma_{\theta\theta}$ increases, but $\sigma_{r\theta}$ decreases at the interface. This implies that the variation of the normal displacement jump becomes large and that variation of the tangential displacement is lower. Since the normal interface displacement jump waves along the interface, a negative jump larger than the interphase thickness means that particles and matrix interpenetrate which is not permissible for physical reasons. It is to be expected that normal imperfect parameters could be different in tension and compression to overcome this paradox. However, in the coated cylinder composite, the amplitude of the $\sigma_{rr}$ and $\sigma_{r\theta}$ all decrease as the imperfect parameters increase, although that of $\sigma_{r\theta}$ increases. This means the variations of the normal and tangential interface displacement jump decrease with the increase of the imperfect parameters and the present model may give a reasonable estimation for the coated fiber composites with imperfect interface.

Figures 5 and 6 show the effect of the interphase properties on the effective shear modulus for the two cases on some imperfect conditions, respectively. It is seen that the two models have a similar effect on the shear modulus of the composites and only the very small interphase properties have a strong influence on the behavior of the composite. This feature is similar to the characteristic of the composites determined with perfect bonding (Wu and Dong, 1995).

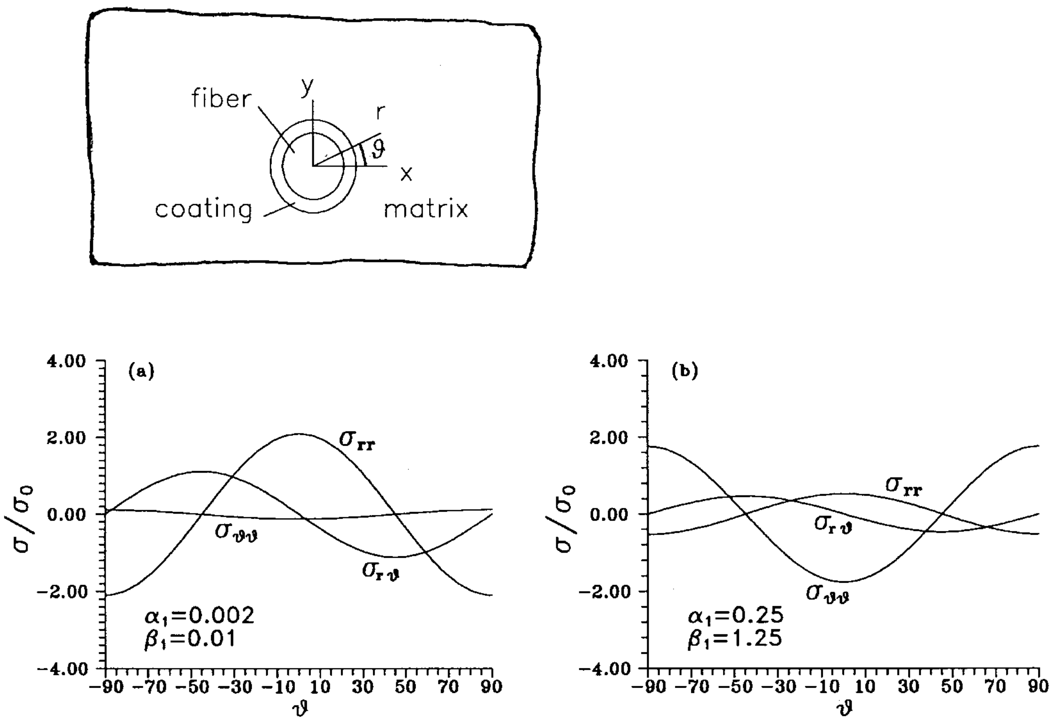


Fig. 4. Stress distributions along the interface between coating and matrix for the shear loading in XY-plane.

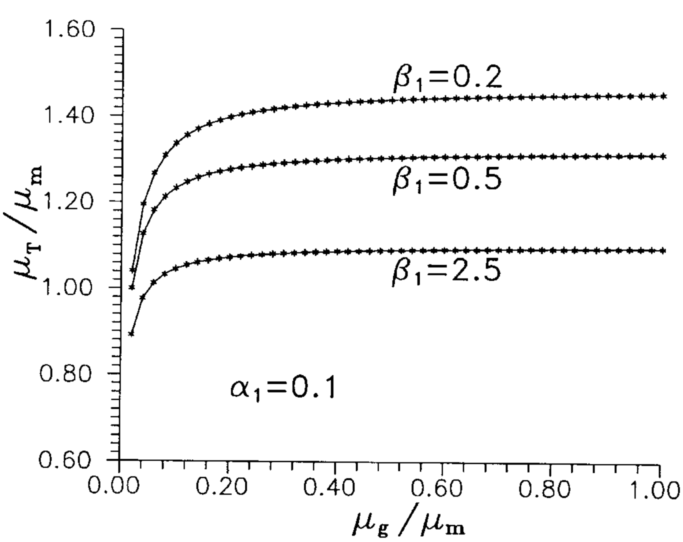


Fig. 5. Variation of effective shear modulus with the interface shear modulus for spherical model.

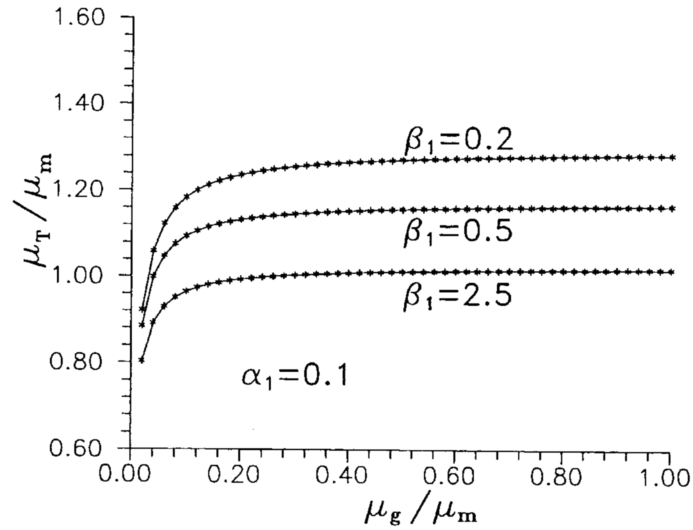


Fig. 6. Variation of effective shear modulus with the interface shear modulus for cylindrical model.

## Acknowledgements

This work was supported by the National Natural Science Foundation of China, Science Foundation of Chinese Academy of Sciences and Laboratory for Nonlinear Mechanics of Continuous Media, Institute of Mechanics, CAS.

## References

Achenbach, J.D., Zhu, H., 1989. Effect of interfacial zone mechanical behavior and failure of fiber-reinforced composites. J. Mech. Phys. Solids 37, 381–393.

Benveniste, Y., 1984. On the effect of debonding on the overall behavior of composite materials. Mech. Mater. 3, 349–368.

Benveniste, Y., 1985. The effective mechanical behavior of composite materials with imperfect contact between the constituents. Mech. Mater. 4, 197–208.

Benveniste, Y., 1987. A new approach to the application of Mori-Tanaka's theory in composite materials. Mech. Mater. 6, 147–157.

Benveniste, Y., Dvorak, G.J., Chen, T., 1989. Stress fields in composite coated inclusions. Mech. Mater. 7, 305–317.

Christensen, R.M., Lo, K.H., 1979. Solutions for effective shear properties in three phase sphere and cylinder models. J. Mech. Phys. Solids 27, 315–329.

Christensen, R.M., Lo, K.H., 1986. Erratum: Solutions for effective shear properties in three phase sphere and cylinder models. J. Mech. Phys. Solids 34, 639.

Hashin, Z., 1991. Thermoelastic properties of particulate composites with imperfect interface. J. Mech. Phys. Solids 39, 745–762.

Mori, T., Tanaka, K., 1973. Average stress in matrix and average elastic energy of materials with misfitting inclusion. Acta Metall. 571–580.

Nemat-Nasser, S., Hori, M., 1993. Micromechanics: Overall Properties of Heterogeneous Material. Elsevier.

Nemat-Nasser, S., Yu, N., Hori, M., 1993. Bound and estimates of overall moduli of composites with periodic microstructure. Mech. Mater. 15, 163–181.

Qu, J., 1993a. Eshelby tensor for an elastic inclusion with slightly weakened interfaces. J. Appl. Mech. 60, 1048–1050.

Qu, J., 1993b. Effects of slightly weakened interfaces on the overall elastic properties of composite materials. Mech. Mater. 14, 269–281.

Walpole, L.J., 1978. A coated inclusion in an elastic medium. Math. Proc. Comb. Phil. Soc. 83, 495–506.

Wu, Yongli, Dong, Zhifa, 1995. Three-dimensional finite element analysis of composites with coated spherical inclusions. Mater. Sci and Engng A 203, 314–323.